A first-order phase transition in a multi-dimensional clustering model

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1 Model description

We investigate a multi-agent model describing the formation of clusters, a phenomenon that can be observed in e.g. swarming behavior of animals, opinion formation, or synchronization in systems of coupled oscillators. Each agent belongs to a multi-dynamical state space, and is characterized by an autonomous component and attraction towards the other agents:

$$\dot{x}_i(t) = b_i + K \sum_{j=1}^{N} \gamma_j f_{ij}(\|x_j(t) - x_i(t)\|) e_{x_j(t)-x_i(t)}, \qquad (1)$$

for all $i \in \{1,\ldots,N\}$, where $\gamma_j > 0$, $K \geq 0$, $N > 1$, and $x_i(t), b_i \in \mathbb{R}^P$. The differentiable functions f_{ij} are non-decreasing with $f_{ij}(0) = 0$, $f_{ij} = f_{ji}$, and $\lim_{\xi \to +\infty} f_{ij}(\xi) = F_{ij}$, for all i and j in $\{1,\ldots,N\}$, for some symmetric matrix $F \in \mathbb{R}^{N\times N}$. Furthermore, $e_x \triangleq \frac{x}{\|x\|}$, for all x in $\mathbb{R}^P \setminus \{0\}$, with $e_0 \triangleq 0$.

2 Clustering behavior

In [2] we show that the long term behavior of the system (1) can be characterized by a set partition H of $\{1,\ldots,N\}$, defining different clusters, such that agents belonging to the same cluster have a common long term average velocity (and agents from different clusters have different long term average velocities). We will refer to this behavior as *clustering behavior* with respect to cluster structure H.

3 Preliminary results for $P = 1$

In [1] we show that, for any choice of the model parameters, the system (1) with $P = 1$ exhibits clustering behavior with respect to some cluster structure H, and we formulate necessary conditions and sufficient conditions (only differing in inequality signs being strict or not) characterizing this cluster structure. Furthermore, if the interaction functions are increasing, then distances between agents from the same cluster approach constant values that are independent of the initial condition.

4 Results for $P > 1$

The case $P > 1$ is harder to analyze, and for the investigation of the emerging cluster structure we focus on two special cases: a system with 3 agents, all-to-all interaction and equal weights, and a system with an infinite number of agents in a spherically symmetric configuration.

In the first case we investigate the transition between a single cluster containing all three agents, and a configuration with three clusters, each containing a single agent, without an intermediate stage with two clusters. For $P > 1$ this may occur for generic values of the model parameters, as opposed to the behavior for the case $P = 1$, where a transition generically involves at most two clusters.

In the second case we calculate a lower and an upper bound for the critical value of the coupling strength corresponding to the origination of a central cluster with velocity zero and containing a non-zero fraction of the population. At this transition value, the central cluster has an initial size different from zero, characteristic of a first-order phase transition.

Acknowledgements

This paper presents research results of the Belgian Network DYSCO (Dynamical Systems, Control, and Optimization), funded by the Interuniversity Attraction Poles Programme, initiated by the Belgian State, Science Policy Office. The scientific responsibility rests with its authors.

During this research Filip De Smet was supported by a Ph.D. fellowship of the Research Foundation - Flanders (FWO).

References

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